Nonlinear balance and asymptotic behavior of supercritical reaction-diffusion equations with nonlinear boundary conditions

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1 Introduction

When one considers reaction diffusion problems with nonlinear boundary (flux) conditions one typically faces problems in which, in a natural way, two different nonlinear mechanisms, of very different nature, compete. Namely, interior reaction and boundary flux. In this context it is therefore a natural question to understand which is the nonlinear balance between these two competing nonlinear mechanisms. From the mathematical point of view a delicate technical problem is also to determine a large class of initial data for which the problem can be solved. Once this is done the balance between the nonlinear terms will determine the subsequent behavior of the solutions.

Let us consider the following problem

$$\left\{\begin{array}{rcll} u_t - \Delta u + f(u) &=& 0 & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \vec{n}} &=& g(u) & \text{on} \quad \Gamma \\ u(0) &=& u_0 & \end{array}\right. \tag{1.1}$$

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in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$. The prototype nonlinearities we consider behave like

$$f(s) \sim c_f |s|^{p-1} s, \qquad g(s) \sim c_g |s|^{q-1} s$$

for $|s| \to \infty$, with $c_f, c_g > 0$. More precisely, we suppose that f and g satisfy

$$pc_f |s|^{p-1} - A_0 \leq f'(s) \leq pC_f |s|^{p-1} + A_1, \qquad qc_g |s|^{q-1} - B_0 \leq g'(s) \leq qC_g |s|^{q-1} + B_1, \tag{1.2}$$

for some $c_f, c_g, C_f, C_g > 0$, $A_0, A_1, B_0, B_1 > 0$, $1 < p, q < \infty$.

These assumptions imply that there is an actual competition in (1.1) between nonlinear terms. On one hand, f has a dissipative character and tries to make solutions global and bounded. On the other hand g has an explosive nature and tries to make solutions blow up in finite time; see [8, 11]. As it is currently known, and explained below, the nonlinear balance between these two terms in (1.1) is given by by the relationship between $p+1$ and $2q$. In fact when

$$p + 1 > 2q \tag{1.3}$$

the dissipative character of f dominates the explosive one of g; see below.

We now review some results already known for (1.1). First, concerning local existence it was shown in [4] that considering initial data in $L^r(\Omega)$, (1.1) is locally well posed provided

$$p \leq p_c = 1 + \frac{2r}{N}, \qquad q \leq q_c = 1 + \frac{r}{N} \tag{1.4}$$

with $q < q_c = 1 + r$, if $N = 1$.

The numbers p_c and q_c above are the so-called critical exponents. The problem (1.1) is said to be subcritical if $p < p_c$ and $q < q_c$, and critical otherwise. See [15, 16, 7] for related results.

It was also proved in [5] that in general supercritical problems, i.e. either $p > p_c$ or $q > q_c$, are ill posed.

As for the asymptotic behavior of solutions, (1.1) is studied in [11] in a subcritical $H^1(\Omega)$ setting (with suitable critical exponents $p_c = 1 + \frac{4}{N-2}$ and $q_c = 1 + \frac{2}{N-2}$ for this space). Assuming the nonlinear balance condition (1.3) and using a natural energy estimate involving the gradient, it was proved that (1.1) is dissipative and the asymptotic behavior of solutions is described by a global compact attractor in $H^1(\Omega)$ which typically has a precise geometrical structure, since the problem has a gradient structure. Note that in this case the energy acts as a Lyapunov functional along solutions, which simplifies a lot the dynamics.

On the other hand if $p+1 < 2q$ the problem is not dissipative and it was shown in [11] that there always exists solutions that blow up in finite time. The case when $p + 1 = 2q$ depends on the balance of the coefficients of the leading or even lower order terms in f and g. See also [2].

In [3], (1.1) was studied in a subcritical regime and assuming a linear balance between nonlinear terms. However for (1.1) under assumptions (1.2) and (1.3) no such linear balance holds.

In [9] the subcritical and critical problems in $L^r(\Omega)$ (i.e, $1 < p \leq p_C$, $1 < q \leq q_c$) were considered. Again under the balance conditions (1.2) and (1.3) it was shown that

(1.1) is dissipative and has well defined asymptotic behavior in terms of a global compact attractor. In the critical case a less conclusive result is obtained. See Theorem 2.1 below for a precise statement.

Note that for the setting in $L^r(\Omega)$, $r \neq 2$, the main difficulty is to obtain gradient estimates to obtain compactness of solutions. In this case no natural energy seems to be available.

Observe that if we assume (1.3) and define

$$r_0 = \max\left\{\frac{N}{2}(p-1), N(q-1)\right\} = \frac{N}{2}(p-1) \tag{1.5}$$

then $r_0 > 1$ if $p > 1 + \frac{2}{N}$ and if $r > r_0$ problem (1.1) is subcritical in $L^r(\Omega)$ while it is critical if $r = r_0$, and supercritical if $1 < r < r_0$. Thus if $r \geq r_0$, given $u_0 \in L^r(\Omega)$ we know that there exists a unique global solution of problem (1.1) and moreover there exists a global compact attractor.

Therefore in this paper we will assume hereafter that

$$p > 1 + \frac{2}{N}$$

and the main goal is the to prove that under assumption (1.3), for the supercritical case $1 < r < r_0$ still we have global bounded solutions and a global compact attractor for (1.1). Moreover this attractor coincides with the one for $r > r_0$.

Since the problem is supercritical in $L^r(\Omega)$ for $1 < r < r_0$ we must then define suitable solutions of (1.1). Then we need to obtain suitable and strong estimates on the solutions which guarantee enough compactness (or smoothing) to prove the existence of an attractor. This estimates will show that solutions of (1.1) enter into spaces in which the problem is subcritical and then they are attracted to the attractor of the subcritical case. As before the main difficulty is to obtain gradient estimates on the solutions.

Our last contribution is to show that within the global attractor there exist extremal equilibria as in [12]. These equilibria have the property that the asymptotic behavior of any solutions lies below the maximal one and above the minimal one. In particular any other equilibria lies in between these two. Also the maximal equilibria is order stable from above and the minimal one is order stable from below. Hence, they are the "caps" of the global attractor. So far such equilibria have been shown to exists in problems like (1.1) but in which a suitable linear balance as in [3] holds; see [12]. For (1.1) under assumptions (1.2) and (1.3) no such linear balance holds and [12] does not apply.

The paper is organized as follows. In Section 2 we recall known results concerning existence of solutions of (1.1) in subcritical or critical cases in $L^r(\Omega)$. Observe that these results only take into account the growth of nonlinear terms and not the signs of f and g. Then, assuming (1.2) and (1.3) we also recall the results on the asymptotic behavior of solutions of (1.1) in subcritical or critical cases; see Theorem 2.1 below.

Then in Section 3 we construct suitable solutions of problem (1.1), assumed (1.2) and (1.3), starting at $u_0 \in L^r(\Omega)$ for $1 < r < r_0$ with r_0 as in (1.5), so that the problem is supercritical in $L^r(\Omega)$. For this, we start by proving the existence and uniqueness of solutions in $L^r(\Omega)$ for an approximated problem in which we truncate the nonlinear term

in the boundary in such a way that the resulting problem is supercritical in the interior while subcritical on the boundary, see Theorems 3.1 and 3.3. Since the supercritical nonlinear term has a good sign, we will be able to obtain suitable uniform bounds for these solutions that will allow us to construct a solution of (1.1). See Propositions 3.5, 3.7 and 3.14, Definition 3.15 and Theorem 3.16. Observe we could only guarantee uniqueness of solutions in the case of nonnegative initial data in (1.1), see Proposition 3.12 but not for general sign changing solutions.

Then in Section 4 we use the strong smoothing estimates obtained in Section 3 to show that all solutions of (1.1), regularize into an space in which (1.1) is subcritical and then the asymptotic behavior of solutions is described by the global attractor in Theorem 2.1. We also show the existence of the extremal equilibria in the attractor as discussed above; see Theorem 4.1. Note that the result in Section 4 improve the known result in Theorem 2.1 for the critical case $r = r_0$.

2 Known results for the subcritical or critical cases

We recall now the results in [4] concerning existence of solutions in subcritical or critical cases in $L^r(\Omega)$. These results only take into account the growth of nonlinear terms and not the signs of f and g. Thus, we assume momentarily that $f, g \in C^1(\mathbb{R})$ and satisfy

$$\limsup_{|s|\to\infty} \frac{|f'(s)|}{|s|^{p-1}} < \infty \quad \text{and} \quad \limsup_{|s|\to\infty} \frac{|g'(s)|}{|s|^{q-1}} < \infty$$

and p, q satisfy (1.4), i.e.

$$p \leq p_c = 1 + \frac{2r}{N}, \qquad q \leq q_c = 1 + \frac{r}{N}, \quad 1 < r < \infty.$$

We will also make use of some Bessel spaces in $L^r(\Omega)$ of order 2θ, $H^{2\theta}_r(\Omega)$; see [4, 1, 14]. These spaces are Bessel spaces associated to Δ with Neumann boundary conditions in $L^r(\Omega)$. Hence, if $2\theta > 1 + \frac{1}{r}$ these spaces incorporate Neumann boundary conditions.

From [4], for each $u_0 \in L^r(\Omega)$, there exist $R = R(u_0) > 0$ and $\tau = \tau(u_0) > 0$ such that for any $u_1 \in L^r(\Omega)$ with $\|u_1 - u_0\|_{L^r(\Omega)} < R$ there exists a solution of (1.1), with initial data u_1, $u(\cdot; u_1)$, in the sense that it is a continuous function $u : [0, \tau_0] \to L^r(\Omega)$ with $u(0) = u_1$, such that $u \in C([0, \tau]; L^r(\Omega)) \cap C((0, \tau]; H^{2\bar{\varepsilon}}_r(\Omega))$ and $\sup_{t\in(0,\tau]} t^{\bar{\varepsilon}} \|u(t)\|_{H^{2\bar{\varepsilon}}_r(\Omega)} < \infty$, for some $\bar{\varepsilon} > 0$ and satisfies the variation of constants formula

$$u(t; u_1) = S(t)u_1 + \int_0^t S(t-s)(-f_\Omega(u(s; u_1)) + g_\Gamma(u(s; u_1)) \, \mathrm{d}s, \quad 0 \leq t \leq \tau,$$

where $S(t)$ is the semigroup generated by Δ with Neumann boundary conditions in $L^r(\Omega)$, f_Ω denotes the Nemitsky map of f acting on functions defined in Ω, and g_Γ the Nemitsky map of g acting on functions defined on Γ.

This is the so called $\bar{\varepsilon}-$regular solution of (1.1) starting at u_1. This solution is unique in the class $C([0, \tau]; L^r(\Omega)) \cap C((0, \tau]; H^{2\bar{\varepsilon}}_r(\Omega))$ and, by a bootstrapping argument, it is classical for $t > 0$.

In addition, this solution satisfies, for some $\underline{\gamma} > \bar{\varepsilon}$ and for all $0 < \theta < \underline{\gamma}$,

$$u \in C((0,\tau]; H_r^{2\theta}(\Omega)), \quad \sup_{t\in(0,\tau]} t^\theta \|u(t)\|_{H_r^{2\theta}(\Omega)} \leq M(R,\tau), \quad t^\theta \|u(t)\|_{H_r^{2\theta}(\Omega)} \xrightarrow{t\to 0^+} 0. \tag{2.1}$$

Moreover, if $u_1, v_1 \in B_{L^r(\Omega)}(u_0, R)$ the following holds true for $t \in (0,\tau]$, and $0 \leq \theta \leq \theta_0 < \underline{\gamma}$,

$$\sup_{t\in(0,\tau]} t^\theta \|u(t;u_1) - u(t;v_1)\|_{H_r^{2\theta}(\Omega)} \leq C(\theta_0,\tau)\|u_1 - v_1\|_{L^r(\Omega)}. \tag{2.2}$$

Note that we can always take $\underline{\gamma} \geq 1/2$ which allows to perform a bootstrap argument to prove that solutions become classical for positive times.

If both of the nonlinearities are subcritical, i.e. $p < p_c$ and $q < q_c$ in (1.4), then R can be taken arbitrarily large and so, the existence time can be taken uniform on bounded sets of $L^r(\Omega)$. As a consequence, and following a standard prolongation argument, when f and g are subcritical in $L^r(\Omega)$, if the solution exists up to a maximal time $T < \infty$ then $\lim_{t\to T} \|u(t)\|_{L^r(\Omega)} = \infty$. However, when f or g are critical, if $T < \infty$ then $\lim_{t\to T} \|u(t)\|_{H_r^\delta(\Omega)} = \infty$ for any $\delta > 0$. Therefore, in the subcritical case to prove global existence it is enough to obtain bounds on the $L^r(\Omega)$–norm of the solution while in the critical case stronger estimates must be obtained.

On the other hand, if we assume now (1.2), that is,

$$pc_f|s|^{p-1} - A_0 \leq f'(s) \leq pC_f|s|^{p-1} + A_1 \quad \text{and} \quad qc_g|s|^{q-1} - B_0 \leq g'(s) \leq qC_g|s|^{q-1} + B_1,$$

for $s \in \mathbb{R}$ and some $c_f, c_g, C_f, C_g > 0$, $A_0, A_1, B_0, B_1 > 0$, and (1.3), i.e.

$$p + 1 > 2q,$$

we can obtain information on the asymptotic behavior of the solutions of the problem.

The next result summarizes the results in [9]. Note that condition (1.3) implies that the dissipative character of f dominates the explosive nature of g. As a result of this nonlinear balance, (1.1) is dissipative as the next theorem shows.

Theorem 2.1 *Assume that f and g satisfy (1.2) and (1.3) and define r_0 as in (1.5). Then we have,*
i) Problem (1.1) is well-posed in $L^r(\Omega)$ for any $r \geq r_0$, and the solutions are globally defined, and classical for $t > 0$.
ii) For $r \geq r_0$, there exists an absorbing ball in $L^r(\Omega)$ and the orbit of any bounded set of $L^r(\Omega)$ is bounded in $L^r(\Omega)$, for $t \geq 0$.

For $r = r_0$, orbits of compact sets in $L^{r_0}(\Omega)$ remain compact in $L^{r_0}(\Omega)$.
iii) If $r > r_0$, (1.1) has a compact global attractor $\mathcal{A}$ in $L^r(\Omega)$ which attracts bounded sets of $L^r(\Omega)$.

For $r = r_0$, there exists a maximal, compact, invariant and connected set $\mathcal{A}$ in $L^{r_0}(\Omega)$ which attracts a neighborhood of each initial data in $L^{r_0}(\Omega)$ (and, in particular, compact sets of $L^{r_0}(\Omega)$).

For $r \geq r_0$ the attractor can be described as the unstable set of the set of equilibria of (1.1), E, which is nonempty; that is

$$\mathcal{A} = W^u(E).$$

iv) For $r \geq r_0$, the attractor $\mathcal{A}$ belongs to $H^\alpha_s(\Omega)$ and it attracts in the norm of $H^\alpha_s(\Omega)$ for any $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{s}$ and in $C^\beta(\overline{\Omega})$ for any $0 \leq \beta < 1$.

In particular, if $r > r_0$ there is an absorbing set in $H^\alpha_s(\Omega)$ and, for every $\varepsilon > 0$, the orbit of bounded sets in $L^r(\Omega)$ is bounded in $H^\alpha_s(\Omega)$ for $t \geq \varepsilon$.

Observe that this result is proved in [9] using in a critical way the energy estimate (2.3) below, which is the only estimate available in a non Hilbertian setting. In [11], (1.1) is studied in a subcritical $H^1(\Omega)$ setting (with suitable critical exponents $p_c = 1 + \frac{4}{N-2}$ and $q_c = 1 + \frac{2}{N-2}$ for this space) using a different energy estimate involving the gradient. For the setting in $L^r(\Omega)$, $r \neq 2$, the main difficulty is to obtain gradient estimates to obtain compactness.

Also observe that the estimates in [9] leading to Theorem 2.1 are not known to be uniform with respect to certain classes of nonlinear terms f and g. Should this be true, some arguments below in Section 3 could be made simpler, see Remark 3.9.

Below we present some of the basic tools needed to prove Theorem 2.1, see [9]. First we have the following Poincaré lemma.

Lemma 2.2 *There exists a constant $c_0(\Omega)$ such that for any $\varphi \in W^{1,1}(\Omega)$*

$$\left\| \varphi - \frac{1}{|\Gamma|} \int_\Gamma \varphi \right\|_{L^1(\Omega)} \leq c_0(\Omega) \|\nabla \varphi\|_{L^1(\Omega)}.$$

Another key tool to prove Theorem 2.1 is the following estimate that holds for any suitable smooth solution of (1.1). This result explains in a precise form the nonlinear balance (1.3) between nonlinear terms in the problem (1.1). Note that the result in [9] is adapted below to (1.1) with f and g satisfying (1.2) and (1.3). We include the proof since it will be important in what follows.

Proposition 2.3 *For a sufficiently smooth solution of (1.1) with f and g satisfying (1.2) and (1.3) we have*

$$\frac{1}{\sigma} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^\sigma_{L^\sigma(\Omega)} + \frac{2(\sigma-1)}{\sigma^2} \int_\Omega \left|\nabla \left(|u|^{\sigma/2}\right)\right|^2 + A \int_\Omega |u|^{\sigma+p-1} \leq B \qquad (2.3)$$

with A depending on the constants appearing on (1.2) but not on σ and $B > 0$ depending also on σ.

Proof. For a sufficiently smooth solution of (1.1), multiplying (1.1) by $|u|^{\sigma-2}u$ and integrating by parts, we get

$$\frac{1}{\sigma} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^\sigma_{L^\sigma(\Omega)} + \frac{4(\sigma-1)}{\sigma^2} \int_\Omega \left|\nabla \left(|u|^{\sigma/2}\right)\right|^2 + \int_\Omega f(u)|u|^{\sigma-2}u - \int_\Gamma g(u)|u|^{\sigma-2}u = 0.$$

The last two terms can be rewritten as

$$\int_\Omega \left(f(u)|u|^{\sigma-2}u - \frac{|\Gamma|}{|\Omega|} g(u)|u|^{\sigma-2}u \right) + \frac{|\Gamma|}{|\Omega|} \int_\Omega \left(g(u)|u|^{\sigma-2}u - \frac{1}{|\Gamma|} \int_\Gamma g(u)|u|^{\sigma-2}u \right)$$

and Lemma 2.2 gives, for some $c(\Omega)$

$$\left|\frac{|\Gamma|}{|\Omega|}\int_\Omega \left(g(u)|u|^{\sigma-2}u - \frac{1}{|\Gamma|}\int_\Gamma g(u)|u|^{\sigma-2}u\right)\right| \le c(\Omega)\|\nabla(g(u)|u|^{\sigma-2}u)\|_{L^1(\Omega)}.$$

Taking derivatives and arranging terms, the right hand side above can be written as

$$c(\Omega)\left\|\left(\frac{2}{\sigma}g'(u)u + \frac{2(\sigma-1)}{\sigma}g(u)\right)|u|^{\sigma/2-1}\left|\nabla|u|^{\sigma/2}\right|\right\|_{L^1(\Omega)}$$

which can be bounded by

$$\varepsilon\|\nabla|u|^{\sigma/2}\|^2_{L^2(\Omega)} + \frac{c^2(\Omega)}{4\varepsilon}\left\|\left(\frac{2}{\sigma}g'(u)u + \frac{2(\sigma-1)}{\sigma}g(u)\right)|u|^{\sigma/2-1}\right\|^2_{L^2(\Omega)}$$

for any $\varepsilon > 0$. Therefore, we get

$$\frac{1}{\sigma}\frac{d}{dt}\|u(t)\|^\sigma_{L^\sigma(\Omega)} + \left(\frac{4(\sigma-1)}{\sigma^2} - \varepsilon\right)\int_\Omega \left|\nabla\left(|u|^{\sigma/2}\right)\right|^2 + \int_\Omega H_\sigma(u)|u|^{\sigma-2} \le 0 \qquad (2.4)$$

where

$$H_\sigma(u) = f(u)u - \frac{|\Gamma|}{|\Omega|}g(u)u - \frac{c^2(\Omega)}{\varepsilon\sigma^2}\left(g'(u)u + (\sigma-1)g(u)\right)^2.$$

From (1.2) we have that,

$$f(u)u \ge D_0|u|^{p+1} - D_1, \quad g(u)u \le D_2|u|^{q+1} + D_1$$
$$(g'(u)u + (\sigma-1)g(u))^2 \le 2(|g'(u)|^2u^2 + (\sigma-1)^2|g(u)|^2) \le D_3(|u|^{2q} + 1)$$

and so,

$$H_\sigma(u) \ge D_0|u|^{p+1} - D_2\frac{|\Gamma|}{|\Omega|}|u|^{q+1} - D_3\frac{c^2(\Omega)}{\varepsilon\sigma^2}|u|^{2q} - D_4$$

for some $D_1, \ldots, D_4 > 0$ where D_0, D_1, D_2 do not depend on σ. Since $p+1 > 2q$ then we have $H_\sigma(u) \ge D_5|u|^{p+1} - D_6$ for some positive constants $D_5 < D_0$ which depends only on p, q and the constants in (1.2) and D_6 depends also in σ and ε. From this we get

$$H_\sigma(u)|u|^{\sigma-2} \ge A|u|^{\sigma+p-1} - B, \quad u \in \mathbb{R}.$$

with A not depending on σ. Then taking $\varepsilon = \frac{2(\sigma-1)}{\sigma^2}$ in (2.4), we get (2.3). ■

Also, we will use below the following Lemma.

Lemma 2.4 *For any smooth enough function in $\overline{\Omega}$, we have, for any $\delta > 0$*

$$\int_\Gamma |\varphi|^r \le \delta\|\nabla(|\varphi|^{r/2})\|^2_{L^2(\Omega)} + c(\Omega, \Gamma, \delta)\|\varphi\|^r_{L^r(\Omega)} \qquad (2.5)$$

Proof. We know that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\int_\Gamma |\xi|^2\,\mathrm{d}x \le \delta\int_\Omega |\nabla\xi|^2\,\mathrm{d}x + C_\delta\int_\Omega |\xi|^2\,\mathrm{d}x$$

for any $\xi \in H^1(\Omega)$. Taking $\xi = |\varphi|^{r/2}$ we obtain (2.5). ■

3 Existence in $L^r(\Omega)$, $1 < r < r_0$

The goal of the section is to prove the existence of a solution of problem (1.1), assumed (1.2) and (1.3), starting at $u_0 \in L^r(\Omega)$ for $1 < r < r_0$ with r_0 as in (1.5), so that the problem is supercritical in $L^r(\Omega)$. For this, we start by proving the existence of solutions in $L^r(\Omega)$ for an approximated problem in which we truncate the nonlinear term in the boundary in such a way that the resulting problem is supercritical in the interior while subcritical on the boundary. Since the supercritical nonlinear term has a good sign, later, suitable uniform bounds for these solutions will allow us to construct a solution of (1.1).

3.1 A supercritical truncated problem

Notice that for f satisfying (3.1) there exists $L > 0$ such that

$$f'(s) \geq -L \qquad \text{for all } s \in \mathbb{R}. \tag{3.1}$$

In fact, it is enough to take $L = A_0$ with A_0 from (1.2).

Also notice that we can construct functions g_K satisfying the following properties

1. $sg_K(s)$ is *increasing* in K, i. e., for any $K_2 > K_1 > 0$,

$$sg_{K_1}(s) \leq sg_{K_2}(s) \leq sg(s) \quad \text{for all } s \in \mathbb{R} \tag{3.2}$$

   and

2. g_K is $C^1(\mathbb{R})$ and

$$-B_0 \leq g'_K(s) \leq K \qquad \text{for all } s \in \mathbb{R}, \tag{3.3}$$

   with B_0 as in (1.2).

3. g_K coincides with g in an interval $I_K = [a_K, b_K]$ with

$$a_K \to -\infty, \quad b_K \to \infty \quad \text{as } K \to \infty. \tag{3.4}$$

Given $K > 0$ we consider the following truncated problem

$$\begin{cases} u_t - \Delta u + f(u) &= 0 \qquad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \vec{n}} &= g_K(u) \quad \text{on} \quad \Gamma \\ u(0) &= u_0 \end{cases} \tag{3.5}$$

with initial data in $L^r(\Omega)$. Notice that the reaction terms in (3.5) satisfy (1.2) and (1.3) with $q = 1$ and

$$sf(s) \geq -|f(0)||s| - L|s|^2, \quad sg_K(s) \leq |g(0)||s| + K|s|^2 \qquad \text{for all } s \in \mathbb{R} \tag{3.6}$$

The next results show that (3.5) is globally well posed in $L^r(\Omega)$. Since (3.5) is supercritical in $L^r(\Omega)$ because $1 < r < r_0$, even local existence does not follow from previous results in Section 2.

Theorem 3.1 *Let $1 < r < r_0$ and $u_0 \in L^r(\Omega)$. Then there exists a function v^K, such that for every $T > 0$,*

$$v^K \in C([0,T]; L^r(\Omega)) \cap L^r((0,T); L^r(\Gamma)), \quad v^K(0) = u_0,$$

and $v^K \in C([\varepsilon, T] \times \overline{\Omega})$, for every $\varepsilon > 0$, with

$$|v^K(t,x)| \leq C(T,K) + t^{-\frac{N}{2r}} C(T,K) \|u_0\|_{L^r(\Omega)}, \qquad 0 < t \leq T \quad \text{for all } x \in \overline{\Omega},$$

for some constant $C(T,K) \geq 0$, which is a global solution of (3.5) in the sense that for all $t \geq 0$ satisfies the variation of constants formula

$$v^K(t) = S(t)u_0 + \int_0^t S(t-s) \left(-f_\Omega(v^K(s)) + (g_K)_\Gamma(v^K(s)) \right) \mathrm{d}s$$

where $S(t)$ denotes the semigroup generated by Δ with Neumann boundary conditions in $L^r(\Omega)$.

Moreover, $v^K \in C^1((0,\infty); C^2(\overline{\Omega}))$ and is a classical solution for $t > 0$ of (3.5).

Proof. We proceed in several steps.

Step 1. Approximate the initial data

Let $u_0^n \in L^\sigma(\Omega)$ for some $\sigma > r_0$, with r_0 as in (1.5), such that $u_0^n \to u_0$ in $L^r(\Omega)$ as $n \to \infty$ and consider the solutions of

$$\begin{cases} u_t - \Delta u + f(u) &= 0 \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{n}} &= g_K(u) \quad \text{on } \Gamma \\ u(0) &= u_0^n \end{cases} \tag{3.7}$$

as in Section 2, which we will denote by $u_n^K(t)$. Since f, g_K satisfy (1.2) and (1.3) with $q = 1$, by part i) in Theorem 2.1, these solutions are global and classical for $t > 0$.

Denote $v_{n,m}^K(t) = u_n^K(t) - u_m^K(t)$. Subtracting equations for u_n^K and u_m^K and multiplying by $|v_{n,m}^K(t)|^{r-2} v_{n,m}^K(t)$ we have

$$\begin{aligned} \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{n,m}^K(t)\|_{L^r}^r + c\|\nabla |v_{n,m}^K(t)|^{r/2}\|_{L^2(\Omega)}^2 \quad &+ \quad \int_\Omega \left[f(u_n^K(t)) - f(u_m^K(t)) \right] |v_{n,m}^K(t)|^{r-2} v_{n,m}^K(t) \\ &\leq \quad \int_\Gamma \left[g_K(u_n^K(t)) - g_K(u_m^K(t)) \right] |v_{n,m}^K(t)|^{r-2} v_{n,m}^K(t). \end{aligned}$$

Now, observe that from (3.1), (3.3)

$$\int_\Omega \left[f(u_n^K) - f(u_m^K) \right] |v_{n,m}^K|^{r-2} v_{n,m}^K \geq -L \|v_{n,m}^K\|_{L^r(\Omega)}^r$$

and

$$\int_\Gamma \left[g_K(u_n^K) - g_K(u_m^K) \right] |v_{n,m}^K|^{r-2} v_{n,m}^K \leq K \int_\Gamma |v_{n,m}^K|^r.$$

Hence, from (2.5), for any $\delta > 0$

$$\int_\Gamma \left[g_K(u_n^K) - g_K(u_m^K)\right] |v_{n,m}^K|^{r-2} v_{n,m}^K \leq \delta \|\nabla(|v_{n,m}^K|^{r/2})\|^2_{L^2(\Omega)} + c(K,\delta)\|v_{n,m}^K\|^r_{L^r(\Omega)}.$$

Thus, with a suitable choice of δ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_{n,m}^K(t)\|^r_{L^r(\Omega)} + c_1\|\nabla(|v_{n,m}^K(t)|^{r/2})\|^2_{L^2(\Omega)} \leq C(K)\|v_{n,m}^K(t)\|^r_{L^r(\Omega)}. \tag{3.8}$$

Given $T > 0$, by Gronwall's Lemma, from (3.8) we have that for any $0 \leq t \leq T$,

$$\|v_{n,m}^K(t)\|^r_{L^r(\Omega)} + c_1 \int_0^t \|\nabla(|v_{n,m}^K(s)|^{r/2})\|^2_{L^2(\Omega)}\,\mathrm{d}s \leq C(K,T)\|v_{n,m}^K(0)\|^r_{L^r(\Omega)} \to 0 \quad \text{as} \quad n,m\to\infty$$

and so, u_n^K is a Cauchy sequence in $C([0,T];L^r(\Omega))$. Also using (2.5)

$$\begin{aligned}\int_0^T \int_\Gamma |v_{n,m}^K(t)|^r \mathrm{d}t &\leq \int_0^T \|\nabla(|v_{n,m}^K(t)|^{r/2})\|^2_{L^2(\Omega)}\mathrm{d}t + c(\Omega,\Gamma)\int_0^T \|v_{n,m}^K(t)\|^r_{L^r(\Omega)}\mathrm{d}t \\ &\leq C(K,T)\|v_{n,m}^K(0)\|^r_{L^r(\Omega)} \to 0 \quad \text{as} \quad n,m\to\infty\end{aligned}$$

and then u_n^K is also a Cauchy sequence in $L^r((0,T);L^r(\Gamma))$.

Hence, there exists $v^K \in C([0,T];L^r(\Omega)) \cap L^r((0,T);L^r(\Gamma))$ such that

$$\sup_{t\in[0,T]} \|u_n^K(t) - v^K(t)\|_{L^r(\Omega)} \to 0 \quad \text{as} \quad n\to\infty \tag{3.9}$$

$$\int_0^T \int_\Gamma |u_n^K(t,x) - v^K(t,x)|^r \,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as} \quad n\to\infty. \tag{3.10}$$

In particular, $u_n^K(t,x) \to v^K(t,x)$ as $n\to\infty$ a.e. for $(t,x) \in [0,T]\times\Gamma$.

Also it is easy to see that v^K does not depend on the sequence of initial data, but only on $u_0 \in L^r(\Omega)$.

Step 2. L^∞-bound for the approaching sequence

Let us show now that the sequence $u_n^K(t)$ is uniformly bounded in $L^\infty(\Omega)$ with respect to n, for $0 < \varepsilon \leq t \leq T$.

For this, we will use the auxiliary problem

$$\left\{\begin{array}{rcll} U_t - \Delta U &=& LU + A & \text{in} \quad \Omega \\ \dfrac{\partial U}{\partial \vec{n}} &=& KU + D & \text{on} \quad \Gamma \\ U(0) & & \text{given in } L^r(\Omega) & \end{array}\right. \tag{3.11}$$

with $A = |f(0)|$ and $D = |g(0)|$. Denote by $U^n(t,x)$ the solution of (3.11) with initial data $|u_0^n|$ and by $U(t,x)$ the solution of (3.11) with initial data $|u_0|$.

Now, using the variation of constants formula in (3.11)

$$U^n(t) = \Phi(t) + U_h^n(t), \quad U(t) = \Phi(t) + U_h(t)$$

where $U^n_h(t), U_h(t)$ are the solutions of the homogeneous problem resulting from taking $A = D = 0$ in (3.11) and initial data $|u^n_0|$ and $|u_0|$ respectively, i.e, the solution of

$$\left\{\begin{array}{rcll} U_t - \Delta U &=& LU & \text{in} \quad \Omega \\ \dfrac{\partial U}{\partial \vec{n}} &=& KU & \text{on} \quad \Gamma \\ U(0) &=& |u^n_0| \quad (\text{or } |u_0|) & \end{array}\right.$$

and $\Phi(t)$ is the unique solution of problem (3.11) with $U(0) = 0$ (which does not depend on u^n_0 or u_0), i.e,

$$\left\{\begin{array}{rcll} U_t - \Delta U &=& LU + A & \text{in} \quad \Omega \\ \dfrac{\partial U}{\partial \vec{n}} &=& KU + D & \text{on} \quad \Gamma \\ U(0) &=& 0. & \end{array}\right.$$

Notice that the homogeneous problem above is a linear heat equation with Robin boundary conditions. Therefore, standard regularity theory implies that U^n_h, U_h and Φ are classical for $t > 0$,

$$\|U^n(t)\|_{L^\infty(\Omega)} \leq C(T,K) + t^{-\frac{N}{2r}} C(T,K) \|u^n_0\|_{L^r(\Omega)}, \qquad 0 < t \leq T,$$

and

$$U^n(t,x) \to U(t,x) \quad \text{in } C^k([\varepsilon, T] \times \Omega) \cap C([0,T]; L^r(\Omega))$$

for any $\varepsilon > 0$, $k \in \mathbb{N}$, since $u^n_0 \to u_0$ in $L^r(\Omega)$ as $n \to \infty$.

Also, $U^n_h(t) \geq 0$, $U_h(t) \geq 0$ and $\Phi(t) \geq 0$ for all $t \geq 0$ since $A, D \geq 0$ and have nonnegative initial data.

Observe now that $U^n(t,x)$ is a supersolution of problem (3.7) since f, g_K satisfy (3.6). Hence

$$|u^K_n(t,x)| \leq U^n(t,x) \leq C(T,K) + t^{-\frac{N}{2r}} C(T,K) \|u^n_0\|_{L^r(\Omega)}, \qquad 0 < t \leq T \quad \text{a.e. in } \Omega,$$

and so $\|u^K_n(t)\|_{L^\infty(\Omega)} \leq C(\varepsilon, T, K, \|u^n_0\|_{L^r(\Omega)})$ for all $n \geq 1$ and $\varepsilon \leq t \leq T$. Also since u^K_n is a classical solution and U^n is also smooth we have that, up to the boundary,

$$|u^K_n(t,x)| \leq U^n(t,x) \leq C(T,K) + t^{-\frac{N}{2r}} C(T,K) \|u^n_0\|_{L^r(\Omega)}, \qquad 0 < t \leq T \quad \text{for all } x \in \overline{\Omega}. \tag{3.12}$$

Now, since $u^n_0 \to u_0$ in $L^r(\Omega)$ as $n \to \infty$ and using the convergence $U^n(t,x) \to U(t,x)$ and $u^K_n(t,x) \to v^K(t,x)$ obtained above, (3.9), (3.10), we get

$$|v^K(t,x)| \leq U(t,x) \leq C(T,K) + t^{-\frac{N}{2r}} C(T,K) \|u_0\|_{L^r(\Omega)}, \qquad 0 < t \leq T \quad \text{for all } x \in \overline{\Omega}. \tag{3.13}$$

Notice that estimates (3.12) and (3.13) are valid up to the boundary.

Finally, observe that the bounds above and (3.9), (3.10) imply that for any $\varepsilon > 0$ and any $r \leq s < \infty$,

$$\sup_{t \in [\varepsilon, T]} \|u^K_n(t) - v^K(t)\|_{L^s(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \tag{3.14}$$

$$\int_\varepsilon^T \int_\Gamma |u^K_n(t,x) - v^K(t,x)|^s \, \mathrm{d}x \mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Step 3. The limit is a solution of (3.5)

First, assume $0 < \varepsilon < t < T$. Taking now $\phi \in H^2_{r'}(\Omega)$, with $\frac{\partial \phi}{\partial \vec{n}} = 0$ on Γ, where r' is the conjugate of r, i. e., $\frac{1}{r} + \frac{1}{r'} = 1$, we have from (3.7)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_n^K \phi + \int_\Omega u_n^K (-\Delta \phi) + \int_\Omega f(u_n^K) \phi = \int_\Gamma g_K(u_n^K) \phi.$$

Now, using the uniform bounds in (3.12), (3.13) and the convergence in (3.14), (3.15), and the growth of f, we have for any $r \leq s < \infty$,

$$f(u_n^K) \to f(v^K) \quad \text{in} \quad L^\infty((\varepsilon, T); L^s(\Omega))$$

and

$$g_K(u_n^K) \to g_K(v^K) \quad \text{in} \quad L^s((\varepsilon, T); L^s(\Gamma)).$$

Hence, letting $n \to \infty$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega v^K \phi + \int_\Omega v^K (-\Delta \phi) + \int_\Omega f(v^K) \phi = \int_\Gamma g_K(v^K) \phi$$

for $\varepsilon \leq t \leq T$.

Then, notice that this is enough to guarantee that v^K satisfies (see [6])

$$v^K(t) = S(t-\varepsilon) v^K(\varepsilon) + \int_\varepsilon^t S(t-s) h(s) \, \mathrm{d}s$$

where $S(t)$ denotes the strongly continuous analytic semigroup generated by Δ in $L^r(\Omega)$ with homogeneous Neumann boundary conditions, and

$$h = -f_\Omega(v^K) + (g_K)_\Gamma(v^K) \in L^1([\varepsilon, T]; L^r(\Omega)) + L^1([\varepsilon, T]; L^r(\Gamma)).$$

Finally, taking $\varepsilon \to 0$ and using the continuity of the linear semigroup $S(t)$ and $v^K(\varepsilon) \to v^K(0) = u_0$ in $L^r(\Omega)$ as $\varepsilon \to 0$, we have that $S(t-\varepsilon)v^K(\varepsilon) \to S(t)u_0$ in $L^r(\Omega)$ as $\varepsilon \to 0$ and then,

$$v^K(t) = S(t) u_0 + \int_0^t S(t-s) h(s) \, \mathrm{d}s.$$

Hence, v^K is a global solution of (3.5) in $L^r(\Omega)$ in the sense of the variations of constants formula.

Step 4. Further regularity

From (3.13), for any $1 < s \leq \infty$ and $\varepsilon > 0$, we have that $v^K(\varepsilon) \in L^s(\Omega)$. Taking $s > r_0$ the problem (3.5) is subcritical in $L^s(\Omega)$. So, by part i) in Theorem 2.1, the unique solution of this problem starting at $v^K(\varepsilon)$, which is $v^K(t+\varepsilon)$, is classical for $t > 0$. Thus, for $t > \varepsilon$, $v^K(t)$ coincides with the solutions in Section 2.

In particular, $v^K \in C([\varepsilon, T] \times \overline{\Omega})$, for every $\varepsilon > 0$, and v^K is a classical solution for $t > 0$ and $v^K \in C((0,T); C^2(\overline{\Omega}))$, for any $T > 0$. ■

Remark 3.2 *If $f(0) = 0 = g(0)$ then $\Phi(t) = 0$ and we can take $C(T, K) = 0$ in (3.13).*

Now we turn into uniqueness of solutions of (3.5).

Theorem 3.3 *Let $1 < r < r_0$ and $T > 0$ fixed. Given $u_0 \in L^r(\Omega)$, there exists a unique function*

$$v \in C([0,T]; L^r(\Omega)) \cap C([\varepsilon, T] \times \overline{\Omega})), \quad v(0) = u_0$$

for any $\varepsilon > 0$, satisfying, for every $0 < \varepsilon \le t \le T$,

$$v(t) = S(t-\varepsilon)v(\varepsilon) + \int_\varepsilon^t S(t-s)\big(- f_\Omega(v(s)) + (g_K)_\Gamma(v(s))\big)\,\mathrm{d}s, \tag{3.16}$$

where $S(t)$ denotes the semigroup generated by Δ with Neumann boundary conditions in $L^r(\Omega)$.

In particular, the function $v^K(\cdot)$ constructed in Theorem 3.1 is the unique function satisfying these conditions.

Proof. Let v^K be the function constructed in Theorem 3.1 and let v be a function such that $v \in C([0,T]; L^r(\Omega))$, $v(0) = u_0$, and for any $\varepsilon > 0$, $v \in C([\varepsilon, T]; C(\overline{\Omega}))$ and satisfies (3.16).

So, fixed $\varepsilon > 0$, $v(\varepsilon) \in C(\overline{\Omega})$ and then from (3.16) and the results in Section 2, v satisfies the equation (3.5) in $[\varepsilon, T]$, with initial data $v(\varepsilon)$, in a classical sense. In particular v is smooth for $t > 0$.

Since both v and v^K satisfy (3.5), arguing as in (3.8) we get, for $0 < \varepsilon \le t \le T$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v^K(t) - v(t)\|^r_{L^r(\Omega)} + c_1\|\nabla(|v^K(t) - v(t)|^{r/2})\|^2_{L^2(\Omega)} \le C(K)\|v^K(t) - v(t)\|^r_{L^r(\Omega)}.$$

Then, by Gronwall's Lemma, we have

$$\|v^K(t) - v(t)\|^r_{L^r(\Omega)} \le C(K,T)\|v^K(\varepsilon) - v(\varepsilon)\|^r_{L^r(\Omega)}, \qquad 0 < \varepsilon \le t \le T.$$

Since $v, v^K \in C([0,T]; L^r(\Omega))$ and $v(0) = v^K(0) = u_0$, taking limits as $\varepsilon \to 0$ we have $v \equiv v^K$ on $[0,T]$. ■

Remark 3.4 *Since the function v in the theorem is continuous at 0 in $L^r(\Omega)$, we can take $\varepsilon \to 0$ in (3.16), to obtain that v satisfies $v(0) = u_0$ and*

$$v(t) = S(t)u_0 + \int_0^t S(t-s)\big(- f_\Omega(v(s)) + (g_K)_\Gamma(v(s))\big)\, ds.$$

Conversely, if we know that the integral above makes sense, then a simple algebraic manipulation implies that (3.16) holds true for any $\varepsilon > 0$.

3.2 Uniform bounds in K for the truncated problem

In order to construct a solution of problem (1.1), we are going to obtain uniform bounds in K for the solutions of (3.5) constructed in Theorem 3.1, for positive times bounded away from zero. We fix $u_0 \in L^r(\Omega)$ and consider the family of solutions $v^K(\cdot; u_0)$ of (3.5) obtained in Theorem 3.1. Then we have the following.

Proposition 3.5 *For any $1 < r < r_0$ and $u_0 \in L^r(\Omega)$, the solutions $v^K(\cdot; u_0)$ of (3.5) obtained in Theorem 3.1 satisfy*

$$\|v^K(t)\|_{L^r(\Omega)} \leq \max\{\|u_0\|_{L^r(\Omega)}, (\beta_r/\gamma_r)^{\frac{1}{r+p-1}}\}, \qquad t \geq 0, \tag{3.17}$$

and for any $\sigma > r$,

$$\|v^K(t)\|_{L^\sigma(\Omega)} \leq \left(\frac{\beta_\sigma}{\gamma_\sigma}\right)^{\frac{1}{\sigma+p-1}} + \left(\frac{\sigma}{\gamma_\sigma(p-1)}\right)^{\frac{1}{p-1}} t^{-\frac{1}{p-1}}, \qquad t > 0 \tag{3.18}$$

for some $\beta_\sigma, \gamma_\sigma > 0$ depending on $\sigma \geq r$ but not in K or u_0.

Finally,

$$\int_0^T \int_\Gamma |v^K|^r \leq CT$$

and for every $0 < \varepsilon < T$ and $\sigma > r$

$$\int_\varepsilon^T \int_\Gamma |v^K|^\sigma \leq CT + \|v^K(\varepsilon)\|^\sigma_{L^\sigma(\Omega)}, \tag{3.19}$$

for some constants depending on $\sigma \geq r$ but not in K or u_0.

Proof.

Step 1. Uniform bounds in K in $C([\varepsilon, \infty); L^\sigma(\Omega))$

We show now that $v^K(t)$ are uniformly bounded, with respect to K, in $L^\sigma(\Omega)$ for all $1 < \sigma < \infty$.

From the regularity of v^K in Theorem 3.1, multiplying the equation in (3.5) by $|v^K|^{\sigma-2}v^K$ we have, see Proposition 2.3,

$$\begin{aligned}\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\|v^K(t)\|^\sigma_{L^\sigma(\Omega)} + c\|\nabla|v^K(t)|^{\sigma/2}\|^2_{L^2(\Omega)} \quad &+ \quad \int_\Omega f(v^K(t))|v^K(t)|^{\sigma-2}v^K(t) \\ &= \quad \int_\Gamma g_K(v^K(t))|v^K(t)|^{\sigma-2}v^K(t).\end{aligned}$$

Since $sg_K(s) \leq sg(s)$ for all $s \in \mathbb{R}$, see (3.2), we have

$$\begin{aligned}\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\|v^K(t)\|^\sigma_{L^\sigma(\Omega)} + c\|\nabla|v^K(t)|^{\sigma/2}\|^2_{L^2(\Omega)} \quad &+ \quad \int_\Omega f(v^K(t))|v^K(t)|^{\sigma-2}v^K(t) \\ &\leq \quad \int_\Gamma g(v^K(t))|v^K(t)|^{\sigma-2}v^K(t).\end{aligned}$$

Finally, proceeding as in (2.3) we have

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{dt}}\|v^K(t)\|^\sigma_{L^\sigma(\Omega)} + \frac{2(\sigma-1)}{\sigma^2}\|\nabla(|v^K(t)|^{\sigma/2})\|^2_{L^2(\Omega)} + A\|v^K(t)\|^{\sigma+p-1}_{L^{\sigma+p-1}(\Omega)} \leq B \tag{3.20}$$

with A depending on the constants appearing on (1.2) but not on σ, K or u_0 and $B>0$ depending on σ but not in K or u_0.

In particular, denoting $y(t)=\|v^K(t)\|^\sigma_{L^\sigma(\Omega)}$ and dropping the gradient term above, we get that $y(t)$ satisfies the following differential inequality

$$\dot{y}(t)+\gamma_\sigma y^{\frac{\sigma+p-1}{\sigma}}(t)\leq \beta_\sigma$$

for some $\beta_\sigma, \gamma_\sigma>0$ not depending on K or u_0.

If $\sigma=r$, since $y(0)<\infty$, we have (3.17). If $\sigma>r$, let $z(t)$ be the solution of

$$\dot{z}+\gamma_\sigma z^{\frac{\sigma+p-1}{\sigma}}=\beta_\sigma$$

with $\lim_{t\to 0^+} z(t)=\infty$. Then from [13], Lemma 5.1, Chapter 3, page 167, we have

$$z(t)\leq\left(\frac{\beta_\sigma}{\gamma_\sigma}\right)^{\frac{\sigma}{\sigma+p-1}}+\frac{1}{\left(\gamma_\sigma\frac{p-1}{\sigma}t\right)^{\frac{\sigma}{p-1}}},\quad t>0.$$

Now, $0\leq y(t)\leq z(t)$ for all $0<t$ and then we get, using $a^\sigma+b^\sigma\leq(a+b)^\sigma$ for $a,b>0$,

$$\|v^K(t;u_0)\|_{L^\sigma(\Omega)}\leq\left(\frac{\beta_\sigma}{\gamma_\sigma}\right)^{\frac{1}{\sigma+p-1}}+\left(\frac{\sigma}{\gamma_\sigma(p-1)}\right)^{\frac{1}{p-1}}\frac{1}{t^{\frac{1}{p-1}}},\quad t>0.$$

Hence, we get (3.18). In particular, we obtain uniform estimates in $L^\sigma(\Omega)$ for $t\geq\varepsilon>0$.

Step 2. Uniform bounds in $L^\sigma([\varepsilon,T];L^\sigma(\Gamma))$

Integrating (3.20) we get for any $0<\varepsilon<T$,

$$\sup_{\varepsilon\leq t\leq T}\|v^K(t)\|^\sigma_{L^\sigma(\Omega)}+c_1\int_\varepsilon^T\left\|\nabla|v^K(s)|^{\sigma/2}\right\|^2_{L^2(\Omega)}\mathrm{d}s+c_2\int_\varepsilon^T\|v^K(s)\|^{\sigma+p-1}_{L^{\sigma+p-1}(\Omega)}\mathrm{d}s\leq c_3T+\|v^K(\varepsilon)\|^\sigma_{L^\sigma(\Omega)} \tag{3.21}$$

for some constants depending on σ but not on K or u_0. Hence, (2.5) and (3.21) give,

$$\int_\varepsilon^T\int_\Gamma|v^K|^\sigma\leq\int_\varepsilon^T\|\nabla(|v^K|^{\sigma/2})\|^2_{L^2(\Omega)}+c\int_\varepsilon^T\|v^K\|^\sigma_{L^\sigma(\Omega)}\leq c_3T+\|v^K(\varepsilon)\|^\sigma_{L^\sigma(\Omega)}.$$

and we get (3.19). ■

Remark 3.6 *A careful analysis of the constants in (3.18), which can be traced back to (3.20) and (2.3), shows that we can not pass to the limit as $\sigma\to\infty$ in (3.18). Therefore with the result above we are not able to obtain $L^\infty(\Omega)$ estimates, uniform in K.*

Having such estimates is very important as we now show.

We now show that a uniform $L^\infty(\Omega)$–bound for $\{v^K\}_K$ it is enough to ensure that any limit of the family $\{v^K\}_K$ of solutions of (3.5) is a classical solution of (1.1) for positive times for which the variations of constant formula holds. We will use this result later to construct a solution of (1.1). In fact we will show later that such uniform $L^\infty(\Omega)$–bound holds true.

Proposition 3.7 *Let $1 < r < r_0$ and $u_0 \in L^r(\Omega)$. Suppose that for any $\varepsilon > 0$ and $T > 0$ the solutions $v^K(\cdot; u_0)$ of (3.5) obtained in Theorem 3.1 satisfy*

$$\|v^K\|_{L^\infty([\varepsilon,T]\times\Omega)} \le C(\varepsilon, T) \tag{3.22}$$

with $C(\varepsilon, T)$ not depending on K. Then, there exists a subsequence of $\{v^K\}_K$, which we denote the same, such that

$$v = \lim_{K\to\infty} v^K \quad in \quad C_{\text{loc}}((0,\infty), H^\alpha_\sigma(\Omega)), \quad v(0) = u_0,$$

for any $r \le \sigma$ and $\alpha < 1 + \frac{1}{\sigma}$. Moreover, for any such subsequence, the limit function satisfies

$$\|v(t)\|_{L^r(\Omega)} \le \max\{\|u_0\|_{L^r(\Omega)}, \left(\frac{\beta_r}{\gamma_r}\right)^{\frac{1}{r+p-1}}\}, \qquad t \ge 0, \tag{3.23}$$

and

$$\|v(t)\|_{L^\sigma(\Omega)} \le \left(\frac{\beta_\sigma}{\gamma_\sigma}\right)^{\frac{1}{\sigma+p-1}} + \left(\frac{\sigma}{\gamma_\sigma(p-1)}\right)^{\frac{1}{p-1}} t^{-\frac{1}{p-1}}, \qquad t > 0, \tag{3.24}$$

with $\beta_\sigma, \gamma_\sigma$ for $\sigma \ge r$ as in (3.18). Also, for any $\varepsilon > 0$

$$\|v(t)\|_{H^\alpha_\sigma(\Omega)} \le C(\varepsilon) \quad for\ all \quad t \ge \varepsilon,$$

with a bound independent of u_0, for any $\sigma > 1$ and $\alpha < 1 + \frac{1}{\sigma}$. Furthermore v satisfies the variation of constants formula

$$v(t) = S(t-\varepsilon)v(\varepsilon) + \int_\varepsilon^t S(t-s)\left(-f_\Omega(v(s)) + g_\Gamma(v(s))\right) ds, \quad t \ge \varepsilon,$$

where $S(t)$ denotes the strongly continuous analytic semigroup generated by Δ in $L^r(\Omega)$ with homogeneous Neumann boundary conditions.

Proof. Observe that from the $L^\infty(\Omega)$ bounds, given $\varepsilon > 0$, there exists $K_0(\varepsilon)$ such that for $K \ge K_0(\varepsilon)$ we have $g_K(v^K(t)) = g(v^K(t))$ for $t \ge \varepsilon$, see (3.4). Then, for $t \ge \varepsilon$, $v^K(t)$ is actually a solution of (1.1) and this solutions lies, for $t \ge \varepsilon$, in a space in which (1.1) is subcritical.

Then Theorem 2.1 with initial data $v^K(\varepsilon)$, implies that, for $K \ge K_0(\varepsilon)$,

$$\|v^K(t)\|_{H^\alpha_\sigma(\Omega)} \le C(\varepsilon) \quad \text{for all} \quad t \ge 2\varepsilon$$

with a bound independent of K and u_0, for any $\sigma > 1$ and $\alpha < 1 + \frac{1}{\sigma}$.

In particular, the bounds above for $t = 2\varepsilon$, imply that, by taking a subsequence (which we will denote the same) if necessary we can assume that $v^K(2\varepsilon)$ converges in $L^\sigma(\Omega)$ for some $\sigma > r_0$. Hence $v(2\varepsilon) = \lim_K v^K(2\varepsilon)$ in $L^\sigma(\Omega)$ with $\sigma > r_0$.

Now, using again the bounds above, the fact that the functions f and g are Lipschitz on bounded sets of $\mathbb{R}$, and that v^K, for $K \geq K_0(\varepsilon)$, satisfies the variations of constants formula

$$v(t) = S(t-2\varepsilon)v(2\varepsilon) + \int_{2\varepsilon}^{t} S(t-s)\left(-f_\Omega(v(s)) + g_\Gamma(v(s))\right) \mathrm{d}s, \quad t \geq 2\varepsilon,$$

where $S(t)$ denotes the strongly continuous analytic semigroup generated by Δ in $L^r(\Omega)$ with homogeneous Neumann boundary conditions, we have that for any $T > 0$ and $\alpha < 1 + \frac{1}{\sigma}$, there exists $L(T)$ such that for $K_1, K_2 \geq K_0(\varepsilon)$, and $t \in [2\varepsilon, T]$,

$$t^\alpha \|v^{K_1}(t) - v^{K_2}(t)\|_{H^{2\alpha}_\sigma(\Omega)} \leq L(T)\|v^{K_1}(2\varepsilon) - v^{K_2}(2\varepsilon)\|_{L^\sigma(\Omega)}$$

as $K_1, K_2 \to \infty$ since the solutions of (1.1) depend continuously on the initial data, see also (2.2).

Therefore, since $\|v^{K_1}(2\varepsilon) - v^{K_2}(2\varepsilon)\|_{L^\sigma(\Omega)} \to 0$, as $K_1, K_2 \to \infty$, we obtain convergence of v^K in $C([\varepsilon, T]; H^\alpha_\sigma(\Omega))$

Hence, taking $\varepsilon_n \to 0$ and using a Cantor diagonal argument, we conclude that there exists a subsequence, that we denote the same $\{v^K\}_K$, such that

$$v = \lim_{K\to\infty} v^K \quad \text{in} \quad C_{\mathrm{loc}}((0,\infty); H^\alpha_\sigma(\Omega)), \quad v(0) = u_0$$

for any $r \leq \sigma$ and $\alpha < 1 + \frac{1}{\sigma}$.

Moreover, from (3.17), (3.18) we get (3.23) and (3.24) . Also, and for any $\varepsilon > 0$

$$\|v(t)\|_{H^\alpha_\sigma(\Omega)} \leq C(\varepsilon) \quad \text{for all} \quad t \geq \varepsilon$$

with a bound independent of u_0, for any $\sigma > 1$ and $\alpha < 1 + \frac{1}{\sigma}$. Moreover, passing to the limit in the variation of constants formula satisfied by v^K, we get that v also satisfies the variation of constants formula

$$v(t) = S(t-2\varepsilon)v(2\varepsilon) + \int_{2\varepsilon}^{t} S(t-s)\big(-f_\Omega(v(\cdot)) + g_\Gamma(v(\cdot))\big) \,\mathrm{d}s, \quad t \geq 2\varepsilon$$

for any $\varepsilon > 0$, where $S(t)$ denotes the strongly continuous analytic semigroup generated by Δ in $L^r(\Omega)$ with homogeneous Neumann boundary conditions. ■

Remark 3.8 *i) Note that as in Remark 3.4, if we knew that the limit function v is continuous at $t = 0$ in $L^r(\Omega)$ then we could take $\varepsilon \to 0$ in the variation of constants formula.*

In fact, below we will prove a weaker form of continuity at $t = 0$ in Theorem 3.16
ii) Note that there might be many limit functions v in the argument in Proposition 3.7.

However, for nonnegative solutions we will show below that the limit function is unique; see Proposition 3.12.

Remark 3.9 *Observe that one could be tempted to follow the following argument: From the bounds on v^K in $L^\sigma(\Omega)$ in Proposition 3.5 for $\sigma > r_0$, which are uniform in $t \geq \varepsilon$, K and u_0, we use Theorem 2.1 in $L^\sigma(\Omega)$ where (1.1) is subcritical and then we obtain the uniform bounds (3.22) in Proposition 3.7.*

However the estimates in Theorem 2.1 are not known to be uniform with respect to the nonlinear terms, which would be changing with K in the argument above.

Hence, to obtain the uniform bounds (3.22) in Proposition 3.7 we use some comparison argument below.

3.3 Positive solutions

We show now that in the case of dealing with positive solutions, we have the bounds (3.22). Moreover in this case we can pass to the limit in K (and not only in a subsequence) and obtain a unique solution of problem (1.1) for which the conclusions of Proposition 3.7 holds.

The tools we use here will also be used to obtain uniform bounds for general sign changing solutions of the problem as we will show in Section 3.4.

Assume that $f(0) \leq 0$ and $g(0) \geq 0$ so that if $0 \leq u_0 \in L^s(\Omega)$ with $s > r_0$ then the solutions of (1.1) and (3.5) are nonnegative for all times, see Appendix A in [3]. Recall that when $s > r_0$ both problems (1.1) and (3.5) are subcritical in $L^s(\Omega)$.

The following result shows in particular that the bounds (3.22) in Proposition 3.7 hold true.

Proposition 3.10 *Assume $f(0) \leq 0$ and $g(0) \geq 0$. Then we have*
i) For any $1 < r < r_0$ and $0 \leq u_0 \in L^r(\Omega)$ the solution $v^K(\cdot; u_0)$ of (3.5) given in Theorem 3.1 is nonnegative for all times.
ii) For any $\varepsilon > 0$, $v^K(t; u_0)$ is bounded in $L^\infty(\Omega)$, uniformly in $t \geq \varepsilon$, K and $u_0 \in L^r(\Omega)$.

Proof. To prove part i) note that it is enough to take $u_0^n \geq 0$ in (3.7) and then $u_n^K(t) \geq 0$ for all $t \geq 0$, which, by the convergence in Theorem 3.1, implies $v^K(t) \geq 0$ for all $t \geq 0$.

Now, to prove part ii), from (3.18) we have that for any $\varepsilon > 0$, $v^K(\varepsilon)$ belongs to a bounded set in $L^\sigma(\Omega)$ which is independent of K and u_0. Taking $\sigma > r_0$ and using that $g_K(s) \leq g(s)$ for $s \geq 0$, see (3.2), we have that

$$0 \leq v^K(t+\varepsilon) \leq u(t; v^K(\varepsilon)), \quad t \geq 0$$

where $u(t; v^K(\varepsilon))$ denotes the solution of (1.1) with initial data $v^K(\varepsilon) \in L^\sigma(\Omega)$, $\sigma > r_0$.

Therefore the dissipativity results in Section 2, see Theorem 2.1, imply that $u(t; v^K(\varepsilon))$ is bounded in $L^\infty(\Omega)$ uniformly in $t \geq \varepsilon$, K and u_0, and so is $v^K(t)$ for $t \geq 2\varepsilon$.

Since $v^K(t)$ is smooth up to the boundary of Ω for $t > 0$, then it is also bounded in $L^\infty(\Gamma)$ uniformly for $t \geq 2\varepsilon$, K and u_0. ■

Remark 3.11 *If $f(0) \geq 0$ and $g(0) \leq 0$, a similar argument gives the bounds on non-negative solutions. In fact now $g_K(s) \geq g(s)$ for $s \leq 0$, and we have that*

$$0 \geq v^K(t+\varepsilon) \geq u(t; v^K(\varepsilon)), \quad t \geq 0$$

where $u(t; v^K(\varepsilon))$ denotes the solution of (1.1) with initial data $v^K(\varepsilon)$.

Now using Proposition 3.7 and the fact that the solutions v^K are nonnegative, we can pass to the limit as $K \to \infty$. Note that below the full family v^K converges and not only a subsequence.

Proposition 3.12 *Assume $f(0) \leq 0$, $g(0) \geq 0$, $1 < r < r_0$ and $0 \leq u_0 \in L^r(\Omega)$. Then for the solutions $v^K(\cdot; u_0)$ of (3.5) given in Theorem 3.1, the limit*

$$v = \lim_{K \to \infty} v^K$$

exists where $v \geq 0$ is as in Proposition 3.7.

Proof. From Proposition 3.10 the functions v^K are nonnegative and from (3.2) $g_K(s)$ is increasing in K for $s \geq 0$. Then it is not difficult to see that for fixed $n \in \mathbb{N}$ the solutions in (3.7) are nonnegative and increasing in K. Therefore the functions v^K in Theorem 3.1 are increasing in K.

Using this and part ii) in Proposition 3.12 we have that the limit

$$0 \leq v(t,x) = \lim_{K \to \infty} v^K(t,x) \quad \text{exists pointwise a.e. in } (0,T) \times \overline{\Omega},$$

and then v coincides with the function v in Proposition 3.7 and the full family v^K converges and not only a subsequence. ■

Remark 3.13 *If $f(0) \geq 0$ and $g(0) \leq 0$, a similar argument allows us to pass to the limit on non-negative solutions (now v^K is decreasing as $K \to \infty$).*

3.4 L^∞ bounds for sign changing solutions

Now we use the arguments in the former subsection to obtain the uniform estimates (3.22) for general sign changing solutions $v^K(\cdot; u_0)$ of (3.5) given in Theorem 3.1.

Proposition 3.14 *Let $1 < r < r_0$ and $u_0 \in L^r(\Omega)$. Then for any $\varepsilon > 0$, $v^K(t; u_0)$ is bounded in $L^\infty(\Omega)$, uniformly in $t \geq \varepsilon$, K and $u_0 \in L^r(\Omega)$.*

Therefore the uniform bounds (3.22) hold and the conclusions of Proposition 3.7 apply.

Proof. From the assumptions on f and g it is clear that we can construct $C^1(\mathbb{R})$ functions f^-, g^+ satisfying (1.2), $g^+(0) \geq 0$ and $g_K(s) \leq g^+(s)$ for all $s \in \mathbb{R}$ and $K > 0$ and $f^-(0) \leq 0$ and $f(s) \geq f^-(s)$ for all $s \in \mathbb{R}$.

Now, from (3.18) we have that for any $\varepsilon > 0$, $v^K(\varepsilon)$ belongs to a bounded set in $L^\sigma(\Omega)$ which is independent of K and u_0. Then we take $\sigma > r_0$ and let $0 \leq U(t; v^K(\varepsilon))$ be the solution of the following problem

$$\left\{ \begin{array}{rcll} U_t - \Delta U + f^-(U) & = & 0 & \text{in } \Omega \\ \dfrac{\partial U}{\partial \vec{n}} & = & g^+(U) & \text{on } \Gamma \\ U(0) & = & |v^K(\varepsilon)| \in L^\sigma(\Omega). & \end{array} \right.$$

Then the comparison principle for subcritical problems, see Appendix A in [3], implies that
$$v^K(t+\varepsilon) \le U(t; v^K(\varepsilon)), \quad t \ge 0.$$

Analogously, we can construct $C^1(\mathbb{R})$ functions f^+, g^- satisfying (1.2), $g^-(0) \le 0$ and $g^-(s) \le g_K(s)$ for all $s \in \mathbb{R}$ and $K > 0$ and $f^+(0) \ge 0$ and $f^+(s) \le f(s)$ for all $s \in \mathbb{R}$.

Then let $W(t; -|v^K(\varepsilon)|) \le 0$ be the solution of the problem

$$\left\{ \begin{array}{rcll} U_t - \Delta U + f^+(U) &=& 0 & \text{in} \quad \Omega \\ \dfrac{\partial U}{\partial \vec{n}} &=& g^-(U) & \text{on} \quad \Gamma \\ U(0) &=& -|v^K(\varepsilon)| \in L^\sigma(\Omega) & \end{array} \right.$$

see Remark 3.11.

Again, the comparison principle for subcritical problems implies

$$W(t; -|v^K(\varepsilon)|) \le v^K(t+\varepsilon) \le U(t; |v^K(\varepsilon)|), \quad t \ge 0.$$

Therefore the dissipativity results in Section 2, see Theorem 2.1, imply that $W(t; -|v^K(\varepsilon)|)$ and $U(t; |v^K(\varepsilon)|)$ are bounded in $L^\infty(\Omega)$ uniformly in $t \ge \varepsilon$, K and u_0, and so is $v^K(t)$ for $t \ge 2\varepsilon$.

Since $v^K(t)$ is smooth up to the boundary of Ω for $t > 0$, then it is also bounded in $L^\infty(\Gamma)$ uniformly for $t \ge 2\varepsilon$, K and u_0. ■

Observe that the results in Propositions 3.7 and 3.14 allows us to define solutions of (1.1) as follows.

Definition 3.15 *For $1 < r < r_0$ and $u_0 \in L^r(\Omega)$ a solution of (1.1) that we denote $u(t; u_0)$ is any of the limit functions v in Proposition 3.7.*

Note in case of nonnegative solutions, from Proposition 3.12, such solution is unique and the same holds for negative solutions, see Remark 3.13.

3.5 Continuity at $t = 0$

In this section we prove that any solution of (1.1) as in Definition 3.15 attains its initial data $u_0 \in L^r(\Omega)$, in compact subsets of Ω but in a slightly weaker norm. In fact we have

Theorem 3.16 *For $1 < r < r_0$ and $u_0 \in L^r(\Omega)$ any solution $u(t; u_0)$, of (1.1) as in Definition 3.15 satisfies for any $1 \le \alpha < r$*

$$u(t; u_0) \to u_0 \quad \text{as } t \to 0, \text{ in } L^\alpha_{loc}(\Omega).$$

Proof. Let Ω_1 be a smooth open subset such that $\overline{\Omega_1} \subset \Omega$ and let $\varphi \in \mathcal{D}(\Omega)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in Ω_1 and $\Omega_0 = supp(\varphi)$ satisfies $\overline{\Omega_1} \subset \Omega_0$ and $\overline{\Omega_0} \subset \Omega$.

Denote then $z^K = v^K \varphi$ which satisfies

$$\left\{ \begin{array}{rcll} z_t - \Delta z &=& -f(v^K)\varphi - 2\nabla v^K \nabla \varphi - v^K \Delta \varphi & \text{in} \quad \Omega_0 \\ z &=& 0 & \text{on} \quad \partial\Omega_0 \\ z(0) &=& u_0 \varphi \in L^r(\Omega_0). & \end{array} \right. \tag{3.25}$$

Now denote $h^K(t) = -f(v^K)\varphi - 2\nabla v^K \nabla\varphi - v^K \Delta\varphi = h_1^K(t) + h_2^K(t) + h_3^K(t)$ and note that from (3.17) we have, for any $T > 0$, $h_3^K \in L^\infty([0,T], L^r(\Omega_0))$, while $h_2^K \in L^\infty([0,T], H^{-1,r}(\Omega_0))$ with norms bounded independent of K.

On the other hand, observe that for any $\alpha \geq 1$ such that $\alpha p > r$ if we take $\sigma > \alpha p$ then we can write

$$\frac{1}{\alpha p} = \frac{\theta}{\sigma} + \frac{1-\theta}{r}, \quad \theta \in (0,1).$$

Then from (3.17) and (3.18) and by interpolation we get, using (1.2),

$$\|f(v^K)\|_{L^\alpha(\Omega)} \leq C(1 + \|v^K\|^p_{L^{\alpha p}(\Omega)}) \leq C(1 + \|v^K\|^{p\theta}_{L^\sigma(\Omega)} \|v^K\|^{p(1-\theta)}_{L^r(\Omega)}).$$

Hence for some C independent of K and for $0 < t < T$,

$$\|f(v^K)\|_{L^\alpha(\Omega)} \leq C\left(1 + \frac{1}{t^{\frac{p\theta}{p-1}}}\right) = C\left(1 + \frac{1}{t^{p'\theta}}\right).$$

Therefore, if $\theta p' < 1$ we have

$$h_1^K \in L^1([0,T], L^\alpha(\Omega_0))$$

with norms bounded independent of K.

Now we show that we can chose α and σ as above. In fact, we have $\frac{1}{\alpha p} = \theta(\frac{1}{\sigma} - \frac{1}{r}) + \frac{1}{r}$ which, using that $\sigma > r$, we write as

$$\frac{p'}{r} - \frac{p'}{\alpha p} = \theta p' \left(\frac{1}{r} - \frac{1}{\sigma}\right).$$

Hence the condition $\theta p' < 1$ can be met provided

$$0 < \frac{1}{\sigma} < \frac{p'}{\alpha p} - \frac{p'-1}{r} = \frac{1}{p-1}\left(\frac{1}{\alpha} - \frac{1}{r}\right)$$

since $\frac{p'}{p} = p' - 1 = \frac{1}{p-1}$. Note that in particular this implies that $\alpha < r$.

On the other hand, using the conditions $\alpha \geq 1$, $\alpha p > r$ and $\sigma > \alpha p$, we are bound to chose α, σ such that

$$0 < \frac{1}{\sigma} < \min\left\{\frac{1}{p-1}\left(\frac{1}{\alpha} - \frac{1}{r}\right), \frac{1}{\alpha p}\right\}, \quad \max\left\{\frac{r}{p}, 1\right\} < \alpha < r.$$

Hence for any $\alpha < r$ there exists such σ.

Now the variations of constants formula for (3.25) gives

$$z^K(t) = S_D(t) u_0 \varphi + \int_0^t S_D(t-s) h^K(s)\, ds$$

where $S_D(t)$ denotes the strongly continuous analytic semigroup generated by Δ with homogeneous Dirichlet boundary conditions.

Now we use that $h^K = h_1^K + h_2^K + h_3^K$ and denote z_1, z_2 and z_3 the corresponding three terms resulting in the integral term above. Then results on linear equations, see e.g.

Theorem 4 in [10], gives that $z_2, z_3 \in C([0,T], L^r(\Omega_0))$ and $z_1 \in C([0,T], L^\alpha(\Omega_0))$ with with norms bounded independent of K, $z_i(0) = 0$ and

$$\sup\{\|z_1(t)\|_{L^\alpha(\Omega_0)}, \|z_2(t)\|_{L^r(\Omega_0)}, \|z_3(t)\|_{L^r(\Omega_0)},\ t \in [0,T]\} \le C(T)$$

with $C(T)$ independent of K and $C(T) \to 0$ as $T \to 0$.

Finally as $S_D(t)u_0\varphi \to u_0\varphi$ in $L^r(\Omega_0)$ as $t \to 0$, we get that for any $\varepsilon > 0$ there exist $\delta > 0$, independent of K, such that

$$\|z^K(t) - u_0\varphi\|_{L^\alpha(\Omega_0)} \le \varepsilon, \quad t \in (0, \delta].$$

In particular, restricting to Ω_1, we get that any $\varepsilon > 0$ there exist $\delta > 0$, independent of K, such that

$$\|v^K(t) - u_0\|_{L^\alpha(\Omega_1)} \le \varepsilon, \quad t \in (0, \delta],$$

which reflects the local equicontinuity at $t = 0$ of the family v^K.

Therefore, passing to the limit along any subsequence as in Proposition 3.7, we get that any solution of (1.1) as in Definition 3.15 satisfies

$$\|v(t) - u_0\|_{L^\alpha(\Omega_1)} \le \varepsilon, \quad t \in (0, \delta]$$

and the result is proved. ∎

4 Asymptotic behavior

Due to the strong smoothing properties obtained above we can actually show now that all solutions of (1.1) as in Definition 3.15 regularize into a space in which (1.1) is subcritical and then the asymptotic behaviour of solutions is described by the global attractor in Theorem 2.1.

In fact, given a bounded set B of initial data in $1 < r < r_0$, from Propositions 3.14 and 3.7 we have that all solutions as in Definition 3.15 starting at B are uniformly bounded in $L^\sigma(\Omega)$ at any time $t \ge \varepsilon$, for any $1 < \sigma < \infty$. Then, for $\sigma > r_0$, problem (1.1) is subcritical and we can use Theorem 2.1.

The following result summarises the results concerning the asymptotic behaviour of solutions of properties (1.1) for initial data in $L^r(\Omega)$ with $1 < r < r_0$.

Theorem 4.1 *Under the assumptions (1.2) and (1.3), for $1 < r < r_0$ we have:*
i) There exists a compact invariant set $\mathcal{A} \subset C^\beta(\overline{\Omega}) \cap H^\alpha_s(\Omega) \subset L^r(\Omega)$, for any $0 \le \beta < 1$, $s \ge 1$ and $0 \le \alpha < 1 + \frac{1}{s}$, attracting the solutions as in Definition 3.15 starting in bounded sets of $L^r(\Omega)$ in the norm of $C^\beta(\overline{\Omega})$ or $H^\alpha_s(\Omega)$.

In particular, there exists an absorbing set in $C^\beta(\overline{\Omega}) \cap H^\alpha_s(\Omega)$. Also $\mathcal{A} = W^u(E)$, that is, the unstable set of the set of equilibria of (1.1), E, which is nonempty. Hence $\mathcal{A}$ is independent of r and coincides with the set in Theorem 2.1.
ii) There exist two extremal equilibria $\varphi_m \le \varphi_{\mathcal{M}}$ for problem (1.1) such that $\varphi_m, \varphi_{\mathcal{M}} \in \mathcal{A}$. Hence any stationary solution φ of (1.1) satisfies

$$\varphi_m(x) \le \varphi(x) \le \varphi_{\mathcal{M}}(x), \quad x \in \Omega.$$

Furthermore,

$$\varphi_m(x) \leq \liminf_{t\to\infty} u(t,x;u_0) \leq \limsup_{t\to\infty} u(t,x;u_0) \leq \varphi_{\mathcal{M}}(x)$$

uniformly in $x \in \Omega$ and for u_0 in bounded sets of $L^r(\Omega)$.

The maximal equilibrium is (order-)stable from above and the minimal one from below.

Proof. Note that in Propositions 3.12 and 3.14 the bounds on the functions v^K are independent of $t \geq \varepsilon$, K and $u_0 \in L^r(\Omega)$. Hence the bounds on $v(t;u_0)$, as in Proposition 3.7 are also independent of $t \geq \varepsilon$ and $u_0 \in L^r(\Omega)$. Then for $t \geq \varepsilon$ all solutions of (1.1) starting at a bounded set of initial data $B \subset L^r(\Omega)$ enter a bounded set in $L^\sigma(\Omega)$ for $\sigma > r_0$. Thus, part i) follows from Theorem 2.1.

Now for part ii) note that from i) we have an absorbing set in $L^\infty(\Omega)$ for all solutions of (1.1). In particular, for some $M > 0$ the ordered interval of functions in Ω such that $-M \leq h(x) \leq M$ for all $x \in \Omega$ in an absorbing interval. Hence we can use Theorem 1.1 in [12] get the existence of two extremal equilibria $\varphi_m, \varphi_{\mathcal{M}} \in L^r(\Omega)$. The maximal equilibrium is (order-)stable from above and the minimal one from below. See also Corollary 3.11 in [12]. ∎

Remark 4.2

i) In particular note that for $r = r_0$ $\mathcal{A}$ attracts bounded sets of $L^{r_0}(\Omega)$ and therefore Theorem 4.1 improves the results in Theorem 2.1 for this critical space.

ii) Restricting only to nonnegative solutions, we have uniqueness for (1.1) as in Proposition 3.12. Using that solutions are smooth after positive time, Theorem 4.5 in [12] allows a more detailed description of the asymptotic behavior of such solutions.

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